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Geometric Phase in a Generalized Time-Dependent Jaynes-Cummings Model

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Abstract By using the Lewis–Riesenfeld invariant theory, we have studied the dynamical and the geometric phases in a generalized time-dependent Jaynes-Cummings model. It is found that the geometric phases in a cycle case have nothing to do with the frequency of the electromagnetic wave, the energy difference between two levels of the atom, and the coupling strength between the atom and the light field.

Keywords Geometric phase · Generalized Jaynes-Cummings model

1 Introduction

It is known that the concept of geometric phase was first introduced by Pancharatnam [1] in studying the interference of classical light in distinct states of polarization. Berry [2] found the quantal counterpart of Pancharatnam's phase in the case of cyclic adiabatic evolution. The extension to non-adiabatic cyclic evolution was developed by Aharonov and Anandan [3]. Samuel et al. [4] generalized the pure state geometric phase further by extending it to non-cyclic evolution and sequential projection measurements. The geometric phase is a consequence of quantum kinematics and is thus independent of the detailed nature of the dynamical origin of the path in state space. This led Mukunda and Simon [5] to put forward a kinematic approach by taking the path traversed in state space as the primary concept for the geometric phase. Further generalizations and refinements, by relaxing the conditions of adiabaticity, unitarity, and cyclicity of the evolution, have since been carried out [6]. Recently, the geometric phase of the mixed states has also been studied [7–9].

As we known that the quantum invariant theory proposed by Lewis and Riesenfeld [10] is a powerful tool for treating systems with time-dependent Hamiltonians. It was generalized

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by introducing the concept of basic invariants and used to study the geometric phases [11–14] in connection with the exact solutions of the corresponding time-dependent Schrödinger equations. The discovery of Berry's phase is not only a breakthrough in the older theory of quantum adiabatic approximations, but also provides us with new insights in many physical phenomena. The concept of Berry's phase has developed in some different directions [15–27]. In this letter, by using the Lewis–Riesenfeld invariant theory, we shall study the dynamical and the geometric phases in a generalized time-dependent Jaynes-Cummings model.

2 Model

It is known that the generalized time-dependent Jaynes-Cummings model describing the interaction of the two-level atoms and the light field associated with the light intensity can be expressed [28]

$$\hat{H} = \omega(t)\hat{a}^\dagger\hat{a} + \frac{\omega_0(t)}{2}\hat{\sigma}_z + g(t)[\sqrt{\hat{a}^\dagger\hat{a}}\hat{a}^\dagger\hat{\sigma}_- + \hat{\sigma}_+\hat{a}\sqrt{\hat{a}^\dagger\hat{a}}], \tag{1}$$

where $\hat{a}$ ($\hat{a}^\dagger$) are the photon annihilation (creation) operators, $\hat{\sigma}_+$ ($\hat{\sigma}_-$) are the pseudospin operators for the atom defined as $\hat{\sigma}_\pm = \hat{\sigma}_x \pm i\hat{\sigma}_y$ with $\hat{\sigma}_x$ and $\hat{\sigma}_y$ being the Pauli matrices, $g(t)$ stands for the coupling strength between the atom and the light field, $\omega(t)$ is the frequency of the electromagnetic wave, and $\omega_0(t)$ is the energy difference between two levels of the atom in the unit $\hbar = 1$.

We can introduce the supersymmetric operators of the system as follows

$$\hat{N} = \hat{a}^\dagger a + \frac{1}{2} = \begin{pmatrix} \hat{a}^\dagger\hat{a} + \frac{1}{2} & 0 \\ 0 & \hat{a}^\dagger\hat{a} + \frac{1}{2} \end{pmatrix}, \qquad \hat{N}' = \begin{pmatrix} (\hat{a}\hat{a}^\dagger)^2 & 0 \\ 0 & (\hat{a}^\dagger\hat{a})^2 \end{pmatrix}, \tag{2}$$

$$\hat{Q} = \sqrt{\hat{a}^\dagger\hat{a}}\hat{a}^\dagger\hat{\sigma}_- = \begin{pmatrix} 0 & 0 \\ \sqrt{\hat{a}^\dagger\hat{a}}\hat{a}^\dagger & 0 \end{pmatrix}, \qquad \hat{Q}^\dagger = \hat{\sigma}_+\hat{a}\sqrt{\hat{a}^\dagger\hat{a}} = \begin{pmatrix} 0 & \hat{a}\sqrt{\hat{a}^\dagger\hat{a}} \\ 0 & 0 \end{pmatrix}. \tag{3}$$

It is easy to find that $\hat{N}$, $\hat{N}'$, $\hat{Q}$ and $\hat{Q}^\dagger$ are the supersymmetric generators and form supersymmetric Lie algebra, namely

$$\hat{Q}^2 = (\hat{Q}^\dagger)^2 = 0, \qquad [\hat{Q}^\dagger, \hat{Q}] = \hat{N}'\hat{\sigma}_z, \qquad [\hat{N}, \hat{N}'] = 0, \qquad [\hat{N}, \hat{Q}] = \hat{Q}, \tag{4}$$

$$[\hat{N}, \hat{Q}^\dagger] = -\hat{Q}^\dagger, \qquad \{\hat{Q}^\dagger, \hat{Q}\} = \hat{N}', \qquad \{\hat{Q}, \hat{\sigma}_z\} = \{\hat{Q}^\dagger, \hat{\sigma}_z\} = 0, \tag{5}$$

$$[\hat{Q}, \hat{\sigma}_z] = 2\hat{Q}, \qquad [\hat{Q}^\dagger, \hat{\sigma}_z] = -2\hat{Q}^\dagger, \qquad (\hat{Q}^\dagger - \hat{Q})^2 = -\hat{N}', \tag{6}$$

where $\{,\}$ stands for the anticommuting bracket. According to (2)–(6), (1) becomes

$$\hat{H} = \omega(t)\hat{N} + \frac{\omega_0(t)}{2}\hat{\sigma}_z + g(t)\hat{Q} + g(t)\hat{Q}^\dagger. \tag{7}$$

It is easy to find that

$$\hat{N}' \begin{pmatrix} |m-1\rangle \\ |m\rangle \end{pmatrix} = m^2 \begin{pmatrix} |m-1\rangle \\ |m\rangle \end{pmatrix}, \tag{8}$$

so we can restrict our study in the sub-Hilbert space of the supersymmetric quasi-algebra $(\hat{N}, \hat{Q}, \hat{Q}^\dagger, \hat{\sigma}_z)$. Below, we replace operator $\hat{N}'$ with the particular eigenvalue m^2.

3 Dynamical and Geometric Phases

For self-consistent, we first illustrate the Lewis-Riesenfeld (L-R) invariant theory [10]. For a one-dimensional system whose Hamiltonian $\hat{H}(t)$ is time-dependent, then there exists an operator $\hat{I}(t)$ called invariant if it satisfies the equation

$$i\frac{\partial \hat{I}(t)}{\partial t} + [\hat{I}(t), \hat{H}(t)] = 0. \tag{9}$$

The eigenvalue equation of the time-dependent invariant $|\lambda_n, t\rangle$ is given

$$\hat{I}(t)|\lambda_n, t\rangle = \lambda_n|\lambda_n, t\rangle, \tag{10}$$

where $\frac{\partial \lambda_n}{\partial t} = 0$. The time-dependent Schrödinger equation for this system is

$$i\frac{\partial |\psi(t)\rangle_s}{\partial t} = \hat{H}(t)|\psi(t)\rangle_s. \tag{11}$$

According to the L-R invariant theory, the particular solution $|\lambda_n, t\rangle_s$ of (11) is different from the eigenfunction $|\lambda_n, t\rangle$ of $\hat{I}(t)$ only by a phase factor $\exp[i\delta_n(t)]$, i.e.,

$$|\lambda_n, t\rangle_s = \exp[i\delta_n(t)]|\lambda_n, t\rangle, \tag{12}$$

which shows that $|\lambda_n, t\rangle_s$ $(n = 1, 2, \ldots)$ forms a complete set of the solutions of (11). Then the general solution of the Schrödinger equation (11) can be written by

$$|\psi(t)\rangle_s = \sum_n C_n \exp[i\delta_n(t)]|\lambda_n, t\rangle, \tag{13}$$

where

$$\delta_n(t) = \int_0^t dt' \langle\lambda_n, t'|i\frac{\partial}{\partial t'} - \hat{H}(t')|\lambda_n, t'\rangle, \tag{14}$$

and $C_n = \langle\lambda_n, 0|\psi(0)\rangle_s$.

For the system described by Hamiltonian (7), we can define the following invariant

$$\hat{I}(t) = \alpha(t)\hat{Q} + \alpha^*(t)\hat{Q}^\dagger + \beta(t)\hat{\sigma}_z. \tag{15}$$

Substituting (7) and (15) into (9), one has the auxiliary equations

$$i\dot{\beta}(t) = m^2 g(t)[\alpha(t) - \alpha^*(t)], \tag{16}$$

$$i\dot{\alpha}^*(t) = -2g(t)\beta(t) + \alpha^*(t)[\omega_0(t) - \omega(t)], \tag{17}$$

$$i\dot{\alpha}(t) = 2g(t)\beta(t) + \alpha(t)[\omega(t) - \omega_0(t)], \tag{18}$$

where dot denotes the time derivative.

Here, we introduce the unitary transformation operator $\hat{V}(t) = \exp[\xi(t)\hat{Q} - \xi^*(t)\hat{Q}^\dagger]$, then it is easy to find that when the following relations hold

$$\sin(2m|\xi(t)|) = \frac{m^2[\alpha(t)\xi^*(t) + \alpha^*(t)\xi(t)]}{2m|\xi(t)|}, \qquad \beta(t) = \cos(2m|\xi(t)|), \tag{19}$$

and

$$\alpha(t)-\frac{\beta(t)\xi(t)}{m|\xi(t)|}\sin(2m|\xi(t)|)+\frac{\xi(t)[\alpha(t)\xi^*(t)+\alpha^*(t)\xi(t)]}{2|\xi(t)|^2}[\cos(2m|\xi(t)|)-1]=0, \quad (20)$$

we can obtain a time-independent invariant as follows

$$\hat{I}_V \equiv \hat{V}^\dagger(t)\hat{I}(t)\hat{V}(t)=\hat{\sigma}_z. \quad (21)$$

According to (19) and (20), we can select

$$\xi(t)=\frac{\theta(t)}{2m}\exp[i\gamma(t)], \qquad \alpha(t)=\frac{\sin\theta(t)}{m}\exp[i\gamma(t)], \quad (22)$$

then the invariant $\hat{I}(t)$ in (15) becomes

$$\hat{I}(t)=\frac{\sin\theta(t)}{m}\{\exp[i\gamma(t)]\hat{Q}+\exp[-i\gamma(t)]\hat{Q}^\dagger\}+\cos\theta(t)\hat{\sigma}_z. \quad (23)$$

On the other hand, in terms of the unitary transformation $\hat{V}(t)$ and the Baker-Campbell-Hausdoff formula [29]

$$\hat{V}^\dagger(t)\frac{\partial\hat{V}(t)}{\partial t}=\frac{\partial\hat{L}}{\partial t}+\frac{1}{2!}\left[\frac{\partial\hat{L}}{\partial t},\hat{L}\right]+\frac{1}{3!}\left[\left[\frac{\partial\hat{L}}{\partial t},\hat{L}\right],\hat{L}\right]+\frac{1}{4!}\left[\left[\left[\frac{\partial\hat{L}}{\partial t},\hat{L}\right],\hat{L}\right],\hat{L}\right]+\cdots, \quad (24)$$

where $\hat{V}(t)=\exp[\hat{L}(t)]$. It is easy to find that when the following two equations hold

$$\frac{\sin\theta(t)\cos\gamma(t)}{2m}[\omega(t)-\omega_0(t)]+g(t)\cos^2\gamma(t)\left[\frac{\theta^2(t)}{2m^2}-1+\cos\theta(t)\right]+g(t)$$
$$-\frac{\dot{\theta}(t)}{2m}\sin\gamma(t)-\frac{\theta(t)}{2m}\dot{\gamma}(t)\cos\gamma(t)+\frac{\dot{\gamma}(t)\cos\gamma(t)}{2m}[\sin\theta(t)-\theta(t)]=0, \quad (25)$$

$$\frac{\sin\theta(t)\sin\gamma(t)}{2m}[\omega(t)-\omega_0(t)]+\frac{g(t)\sin 2\gamma(t)}{2}\left[\frac{\theta^2(t)}{2m^2}-1+\cos\theta(t)\right]$$
$$-\frac{\dot{\theta}(t)}{2m}\cos\gamma(t)+\frac{\theta(t)}{2m}\dot{\gamma}(t)\sin\gamma(t)+\frac{\dot{\gamma}(t)\sin\gamma(t)}{2m}[\sin\theta(t)-\theta(t)]=0, \quad (26)$$

one has

$$\hat{H}_V(t)=\hat{V}^\dagger(t)\hat{H}(t)\hat{V}(t)-i\hat{V}^\dagger(t)\frac{\partial\hat{V}(t)}{\partial t}=\omega(t)\hat{N}+\left\{\frac{\omega(t)}{2}[1-\cos\theta(t)]\right.$$
$$\left.+\frac{\omega_0(t)}{2}\cos\theta(t)+mg(t)\sin\theta(t)\cos\gamma(t)\right\}\hat{\sigma}_z+\frac{\dot{\gamma}(t)}{2}[1-\cos\theta(t)]\hat{\sigma}_z. \quad (27)$$

The eigenstates of operator $\hat{\sigma}_z$ corresponding to the eigenvalues $\sigma=+1$ and $\sigma=-1$ are $\binom{1}{0}$ and $\binom{0}{1}$, and the eigenstate of operator $\hat{N}'$ is $\binom{|m-1\rangle}{|m\rangle}$ from (8), then we can obtain two particular solutions of the time-dependent Schrödinger equation of the system, i.e., for $\sigma=+1$,

$$|\psi^{m-1}_{\sigma=+1}(t)\rangle=\exp\left\{-i\int_0^t[\dot{\delta}^d_{\sigma=+1}(t')+\dot{\delta}^g_{\sigma=+1}(t')]dt'\right\}\hat{V}(t)\begin{pmatrix}|m-1\rangle\\0\end{pmatrix}, \quad (28)$$

where

$$\dot{\delta}^d_{\sigma=+1}(t') = m\omega(t') + \frac{\omega_0(t') - \omega(t')}{2}\cos\theta(t') + mg(t')\sin\theta(t')\cos\gamma(t'), \tag{29}$$

$$\dot{\delta}^g_{\sigma=+1}(t') = \frac{\dot{\gamma}(t')}{2}[1 - \cos\theta(t')], \tag{30}$$

and for $\sigma = -1$,

$$|\psi^m_{\sigma=-1}(t)\rangle = \exp\left\{-i\int_0^t [\dot{\delta}^d_{\sigma=-1}(t') + \dot{\delta}^g_{\sigma=-1}(t')]dt'\right\}\hat{V}(t)\begin{pmatrix} 0 \\ |m\rangle \end{pmatrix}, \tag{31}$$

where

$$\dot{\delta}^d_{\sigma=-1}(t') = m\omega(t') + \frac{\omega(t') - \omega_0(t')}{2}\cos\theta(t') - mg(t')\sin\theta(t')\cos\gamma(t'), \tag{32}$$

$$\dot{\delta}^g_{\sigma=-1}(t') = -\frac{\dot{\gamma}(t')}{2}[1 - \cos\theta(t')]. \tag{33}$$

From (28)–(33), we conclude that the dynamical and the geometric phase factors of the system are $\exp[-i\int_0^t \dot{\delta}^d_\sigma(t')dt']$ and $\exp[-i\int_0^t \dot{\delta}^g_\sigma(t')dt']$ with $\sigma = \pm 1$, respectively. In particular, when we consider a cycle in the parameter space of the invariant $\hat{I}(t)$ and let $\theta(t)$=constant, then (30) and (33) becomes respectively,

$$\delta^g_{\sigma=+1}(T) = \frac{1}{2}2\pi(1 - \cos\theta), \qquad \delta^g_{\sigma=-1}(T) = -\frac{1}{2}2\pi(1 - \cos\theta), \tag{34}$$

here $2\pi(1 - \cos\theta)$ denotes the solid angle over the parameter space of the invariant $\hat{I}(t)$. It is pointed out that the geometric phases in the cycle case have nothing to do with the frequency $\omega(t)$ of the electromagnetic wave, the energy difference $\omega_0(t)$ between two levels of the atom in the unit $\hbar = 1$, and the coupling strength $g(t)$ between the atom and the light field.

4 Conclusions

In this letter, we have studied the dynamical and the geometric phases in a generalized time-dependent Jaynes-Cummings model. We find that, different from the dynamical phases, the geometric phases in a cycle case have nothing to do with the frequency of the electromagnetic wave, the energy difference between two levels of the atom, and the coupling strength between the atom and the light field.

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